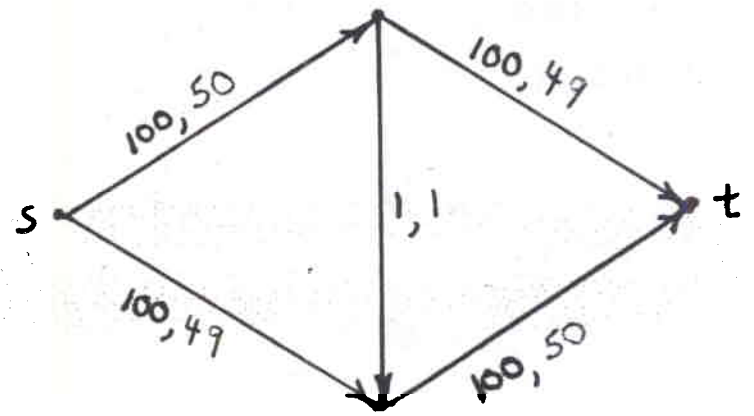
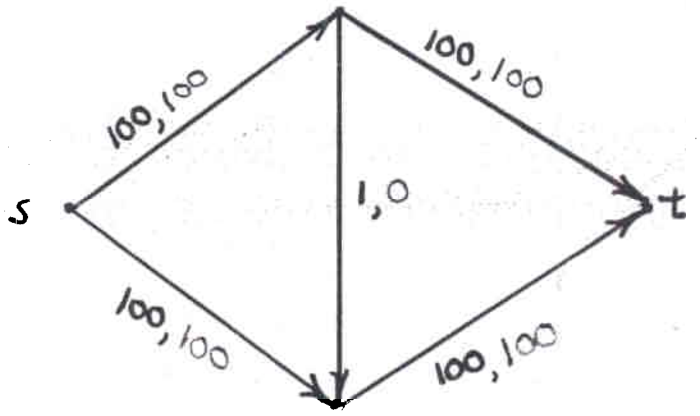
Maximum Network Flow

Network: a directed graph, with two distinguished vertices, a source s and a sink t, and a positive capacity $u(v,w)$ on each edge (v,w).

A flow on a network: a nonnegative function f on edges, bounded above by the capacities, such that the total flow into any vertex other than s and t equals the total flow out



Maximum flow: a flow that maximizes the net flow into the sink (which equals the net flow out of the source).



Problem: Find a maximum flow in a given network, as fast as possible.

n = # vertices

m = # edges

U = maximum edge capacity
(if capacities are integers)

Ford- Fulkerson Method

Residual edge: a pair (v, w) such that

$$\text{(i)}\ f(v,w) < u(v,w):\ u_f(v,w) = u(v,w) - f(v,w)$$

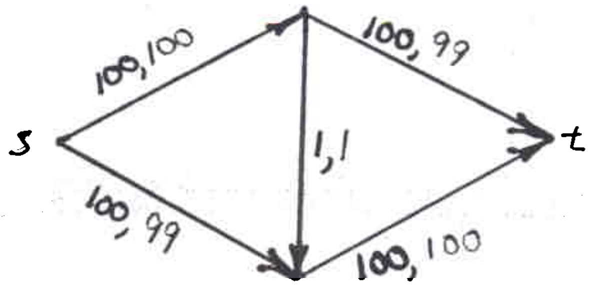
or

$$\text{(ii)}\ f(w,v) > 0:\ u_f(v,w) = f(w,v)$$

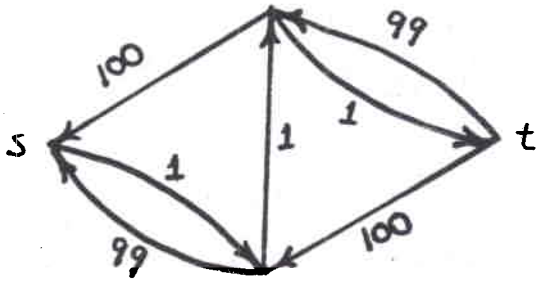
Residual network: the network of residual edges

Thm. A flow is maximum iff there is no path from s to t in the residual network (such a path is an augmenting path).

Network



Residual Network



Ford-Fulkerson method:

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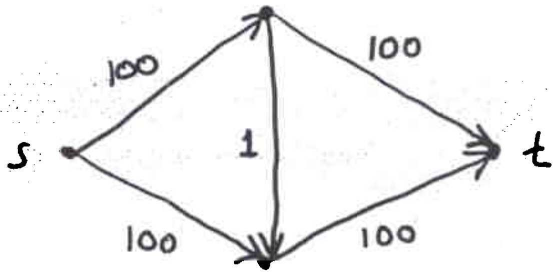
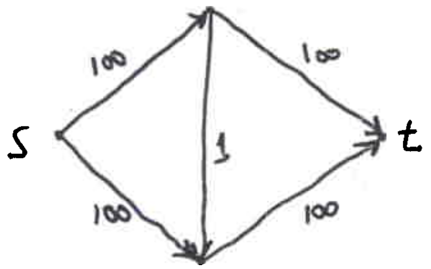
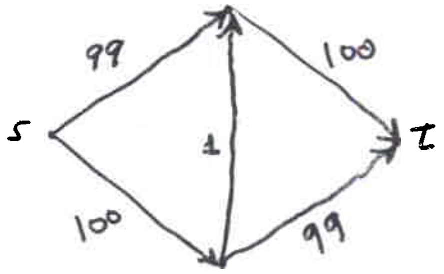
repeat {
    find an augmenting path
    augment flow
}
```

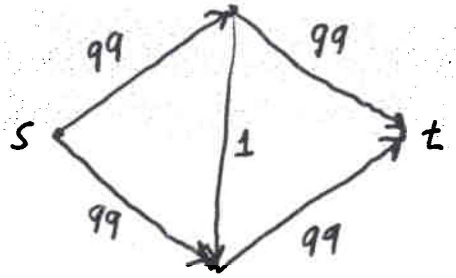
Time: $O(nmU)$ (not polynomial, need not terminate if capacities are irrational)

(Bad) Example









etcetera

Maximum Flow Problem

Network $G=(V,E)$, source s, sink t

edge capacities $u(v,w)$ for $(v,w) \in E$

$|V| = n \quad |E| = m \quad U = \max |u(v,w)|$

Assume network is symmetric:

$(v,w) \in E$ iff $(w,v) \in E$

Flow $f : E \rightarrow \mathbb{R}$

$$f(v,w) \leq u(v,w)$$

$$f(v,w) = -f(w,v)$$

$$e(w) = \sum_{w} f(v,w) = 0 \qquad \forall w \notin \{s,t\}$$

Objective: maximize $e(t)$ $(= -e(s))$

Edmonds & Karp: augment along shortest (fewest edges) paths: $O(nm^2)$

Dinits: build shortest path subnetwork of residual network, find all augmenting paths of a given length at once: $O(n^2m)$

An edge (v,w) is saturated if $f(v,w) = u(v,w)$

A blocking flow is a flow such that every path from s to t contains a saturated edge

Dinits reduced the maximum flow problem to n blocking flow problems, each on an acyclic network.

Finding a blocking flow is easier than finding a maximum flow, at least on an acyclic network.

Edmonds & Karp: always augment along a shortest (fewest edges) path:

$O(m)$ time per path × $O(m)$ paths per length × $O(n)$ path lengths = $O(nm^2)$ time

Dinic: find all augmenting paths of a given length at once, in a phase:

$O(n)$ time per path × $O(nm)$ paths + $O(m)$ time per phase × $O(n)$ phases = $O(n^2m)$ time

Classical Algorithms

Date	Discoverer	Time
1956	Ford & Fulkerson	$O(nmU)$
1969	Edmonds & Karp	$O(nm^2)$
1970	Dinic	$O(n^2m)$
1974	Karzanov (same bound by several others later)	$O(n^3)^*$
1977	Cherkasky	$O(n^2m^{1/2})^*$
1978	Galil	$O(n^{5/3}m^{2/3})^*$
1978	Galil & Naamad; Shiloach	$O(nm(\log n)^2)$
1980	Sleator & Tarjan	$O(nm \log n)$
1983	Gabow	$O(nm \log U)$

*Forerunners of preflow push method

Techniques

Iterative Improvement:
locally modify the current solution to improve it

Successive Approximation:
solve successively closer approximations of the original problem, using each solution as a starting point for the next problem

Data Structures:
represent relevant information about the current flow in an appropriate way.

Preflow Push Approach (Goldberg)

Two ideas:

Make the basic steps in the computation smaller (relax the flow conservation requirement)

Use a less global, more distributed approach to do the preprocessing associated with each phase

Main effect: simpler algorithms

Preflow (Karzanov): like a flow except that the total flow into a vertex can exceed the total flow out.

A vertex $\neq t$ with extra incoming flow is active. The net incoming flow $e(v)$ is the excess of vertex v.

Idea: move flow excess toward sink along estimated shortest paths. Move excess that cannot reach the sink back to the source, also along estimated shortest paths.

To estimate path lengths: a valid labeling is an integer function d on vertices such that:

(i) $d(t) = 0$
(ii) $d(s) = n$
(iii) $d(v) \leq d(w) + 1$ if $u_f(v,w) > 0$

$d(v)$ is a lower bound on the minimum of distance to t, $n +$ distance to s

Algorithm

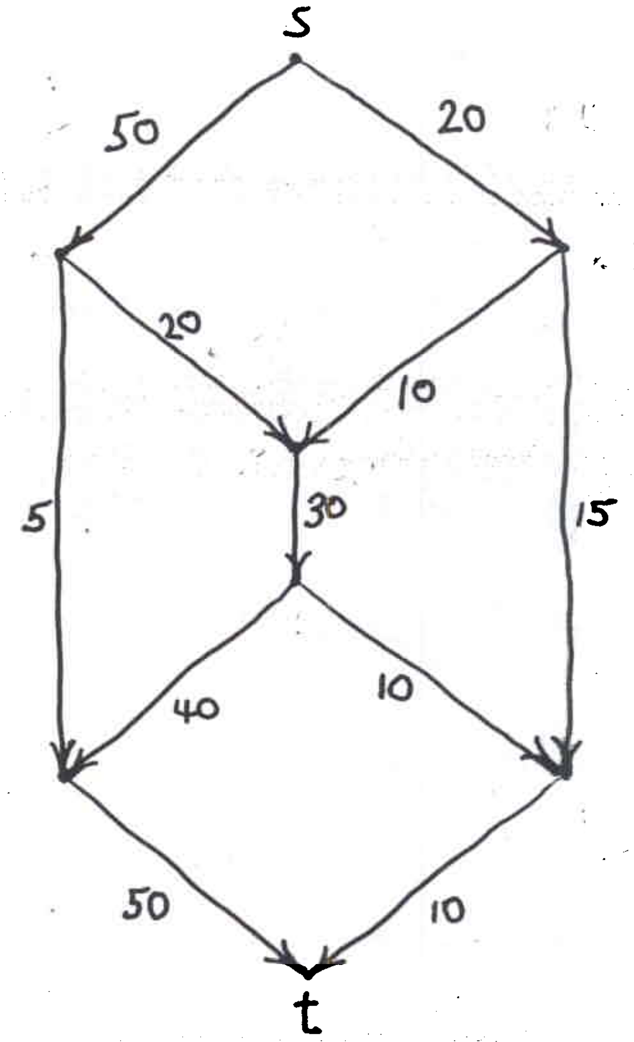
1. Saturate all edges leaving s. Choose initial d.
2. Repeat push and relabel steps in any order until no vertex is active.

push(v,w):

if v is active, $u_f(v,w) > 0$, and $d(v) = d(w) + 1$ then move up to $\min\{e(v), u_f(v,w)\}$ units of flow from v to w (the push is <u>saturating</u> if $u_f(v,w)$ units are moved)

relabel(v):

if v is active and for all (v,w), $u_f(v,w) = 0$ or $d(v) \leq d(w)$ then let $d(v) = \min\{d(w) + 1 \mid u_f(v,w) > 0\}$


s
50
20
20
10
5
30
15
40
10
50
10
t

Bounds

Every active vertex has a label of at most $2n-1$:

there is always a residual path to s.

$\Rightarrow O(n^2)$ relabelings, taking $O(nm)$ time.

Between saturating pushes through the same edge, ends

of edge must be relabeled

$\Rightarrow O(nm)$ saturating pushes.

The heart of the analysis is in bounding

the number of nonsaturating pushes.

Generic Bound: $O(n^2 m)$

Pf. Define $\Phi = \sum_{v \text{ active}} d(v)$.

$0 \le \Phi \le 2n^2$. A nonsaturating push decreases Φ by one.

Increases to Φ: $O(n^2)$ in total due to relabelings.

$O(n^2 m)$ due to saturating pushes:

$O(n)$ per saturating push.

$\Rightarrow O(n^2 m)$ nonsaturating pushes.

FIFO Method

Maintain a queue of active vertices.

Always push from the vertex on the front of the queue.

Add newly active vertices to the rear of the queue.

Analysis

Phases: phase 1 = processing of vertices originally on queue.

phase i+1 = processing of vertices added to queue during phase i.

Only one nonsaturating push per vertex per phase: such a push reduces the excess to zero and removes the vertex from the queue.

$O(n^2)$ bound on # phases

Define $\Phi = \max_{v \text{ active}} d(v)$. $\quad 0 \le \Phi \le 2n$.

A phase reduces Φ by one unless a relabeling occurs.

All increase in Φ is due to relabelings, totals $O(n^2)$.

The number of phases in which Φ doesn't change is also $O(n^2)$.

$\Rightarrow$ $O(n^2)$ total phases.

$\Rightarrow$ $O(n^3)$ nonsaturating pushes.

Ahuja-Orlin Excess Scaling

Maintain Δ, an upper bound on max excess

Maintain integrality of flow.

After each phase, replace Δ by $\Delta/2$.

Stop when $\Delta < 1$.

Push from a vertex v of smallest $d(v)$ with

$e(v) > \Delta/2$.

When pushing from v to $w \neq t$, move

$$\min\{e(v), u_f(v,w), \Delta - e(w)\}$$

Analysis

Each nonsaturating push moves at least

$\Delta/2$ units of flow.

Let $\Phi = \sum_{v \text{ active}} e(v)\,d(v)/\Delta$

$0 \le \Phi \le 2n^2$

Each nonsaturating push decreases Φ by $\ge 1/2$.

Increases in Φ: $O(n^2)$ associated with relabeling.

$O(n^2)$ per phase from change in Δ.

$O(\log U)$ phases $\Rightarrow$

$O(n^2 \log U)$ nonsaturating pushes

saturating pushes = $O(nm)$

nonsaturating pushes = $O(n^2 \log U)$

Can these estimates be balanced ?

Yes: change algorithm : make <u>all</u> pushes large enough by retaining enough excess to immediately saturate very-small-capacity edges.

\# pushes = $O\left(n^{3/2} m^{1/2} (\log U)^{1/2}\right)$

Cheriyan - Mehlhorn

What about relabeling time ??

Preflow Push Algorithms

Date	Discoverer	Time	Method
1985	Goldberg	$O(n^3)$	FIFO
1987	Cheriyan & Maheshwari	$O(n^2 m^{1/2})$	Max Distance
1986	Goldberg & Tarjan	$O(nm \log(\frac{n^2}{m}))$	FIFO + Trees
1986	Ahuja & Orlin	$O(nm + n^2 \log U)$	Excess Scaling
1987	Ahuja & Orlin	$O(nm + n^2 (\log U)^{1/2})$	"
1987	Ahuja, Orlin, & Tarjan	$O(nm \log(\frac{n}{m}(\log U)^{1/2} + 2))$	Excess Scaling + Trees
?	?	$O(nm)$	?
1989	Cheriyan & Hagerup (improved)	$O(nm + n^2 (\log n)^2)$	Excess Scaling + Trees + Randomization
1989	Cheriyan & Hagerup + Mehlhorn	$\sim O(n^3/\log n)$	+ Random Access
?	?	?	?

Practice

Appropriate versions of the preflow push method are easy to implement and very fast in practice: 4-14 times faster than Dinic on reasonable classes of graphs.

Important heuristic: periodically compute tight distance labels using breadth-first search. (Otherwise the relabeling time is too high.)

The FIFO algorithm can be parallelized: push from all active vertices at once. It seems to give drastic speedups in practice.

Whether dynamic trees help on very large graphs has not yet been studied.